Calculus Summary Sheet

**Limits**

**Trigonometric Limits:**

\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1, \quad \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0
\]

**Squeeze Theorem:** If \( f(x) \leq g(x) \leq h(x) \) and if

\[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \text{ then } \lim_{x \to a} g(x) = L
\]

**Indeterminate Forms:**

\[
0 \cdot \infty, \quad \infty - \infty, \quad 0 \cdot 0, \quad 1^\infty, \quad \infty^0
\]

**L’Hôpital’s Rule:** If \( f(x) \) and \( g(x) \) are two differentiable functions, if

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = 0 \quad \text{or} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \infty
\]

Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

**Derivatives**

**Common Derivatives:**

\[
\frac{d}{dx} x^n = nx^{n-1}, \text{ } n \text{ is a real number and } n \neq 0.
\]

\[
\frac{d}{dx} e^x = e^x \quad \frac{d}{dx} \ln(x) = \frac{1}{x}, \quad x > 0 \quad \frac{d}{dx} a^x = a^x \ln a, \quad a > 0
\]

\[
\frac{d}{dx} \log_a(x) = \frac{1}{x \ln a}, \quad x > 0 \text{ and } a > 0
\]

\[
\frac{d}{dx} \sin(x) = \cos(x) \quad \frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}
\]

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\[
\frac{d}{dx} \cos(x) = -\sin(x) \quad \frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}
\]

\[
\frac{d}{dx} \tan(x) = \sec^2(x) \quad \frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}
\]

\[
\frac{d}{dx} \sec(x) = \sec(x) \tan(x) \quad \frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}
\]

\[
\frac{d}{dx} \cot(x) = -\csc^2(x) \quad \frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1+x^2}
\]

\[
\frac{d}{dx} \csc(x) = -\csc(x) \cot(x) \quad \frac{d}{dx} \csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}
\]

\[
\frac{d}{dx} \cosh(x) = \sinh(x) \quad \frac{d}{dx} \sinh(x) = \cosh(x)
\]

\textbf{Chain Rule:} \quad \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)

\textbf{Product Rule:} \quad \frac{d}{dx} [f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)

\textbf{Quotient Rule:} \quad \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}

\textbf{Chain Rule for } e^x: \quad \frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)

\textbf{Chain Rule for } \ln x: \quad \frac{d}{dx} \ln(g(x)) = \frac{g'(x)}{g(x)}

\textbf{Increase/Decrease, Relative Max/Min, Concavity}

\bullet \text{If } f'(x) > 0 \text{ on } (a, b), \text{ then } f(x) \text{ is increasing on } (a, b).

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• If \( f'(x) < 0 \) on \((a, b)\) then \( f(x) \) is **decreasing** on \((a, b)\).

• If \( f'(x) = 0 \) on \((a, b)\), then \( f(x) \) is **constant** on \((a, b)\).

• A number \( c \) in the domain of \( f(x) \) is a **critical point** if: \( f'(c) = 0 \) or \( f'(c) \) is undefined.

• If \( c \) is a critical point of \( f(x) \) then:
  
  * If \( f'(x) \) changes sign from positive to negative (+ to -) at \( c \), then \( f(c) \) is a **local maximum**.
  
  * If \( f'(x) \) changes sign from negative to positive (- to +) at \( c \), then \( f(c) \) is a **local minimum**.

• If \( f''(x) > 0 \) on \((a, b)\), then \( f(x) \) is **concave up** on \((a, b)\).

• If \( f''(x) < 0 \) on \((a, b)\), then \( f(x) \) is **concave down** on \((a, b)\).

• If \( f''(c) = 0 \) or \( f''(c) \) is undefined for some point \( c \) in \((a, b)\), \( f(x) \) is **continuous at** \( c \) and \( f''(x) \) changes sign at \( c \), then \( c \) is a **point of inflection**.

**Second Derivative Test** If \( c \) is a critical point and \( f''(c) \) exists then we have the following:

• \( f''(c) > 0 \), then \( f(c) \) is a **local minimum**.

• \( f''(c) < 0 \), then \( f(c) \) is a **local maximum**.

• \( f''(c) = 0 \), then the test is **inconclusive**. Try looking at at \( f'(x) \).

**Mean Value Theorem:** If \( f(x) \) is a continuous function on a closed interval \([a,b]\) and differentiable on the interval \((a,b)\) then there exists a number \( c \) in \((a,b)\) such that:

\[
\frac{f(b)-f(a)}{b-a} = f'(c)
\]
**Rolle’s Theorem:** If $f(x)$ is continuous on an interval $[a, b]$, differentiable on an interval $(a, b)$, and $f(a) = f(b)$, then there exists a number $c$ in $(a, b)$ such that $f'(c) = 0$

**Newton’s Method** To find an approximation to $f(x) = 0$:

1. Choose $x_0$, your initial guess. Make sure your initial guess is not a point where $f'(x) = 0$ or close to that!!
2. Use the following formula to give you the approximations:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**Integrals**

Common Antiderivatives (Integrals):

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1.$$  

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int \tan(x) \, dx = -\ln|\cos(x)| + C$$

$$\int \cot(x) \, dx = \ln|\sin(x)| + C$$

$$\int \sec(x) \, dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x) \, dx = \ln|\csc(x) - \cot(x)| + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + C$$

$$\int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1}(x) + C$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + C$$

$$\int \frac{-1}{1+x^2} \, dx = \cot^{-1}(x) + C$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} \, dx = \sec^{-1}(x) + C$$

$$\int \frac{-1}{|x|\sqrt{x^2-1}} \, dx = \csc^{-1}(x) + C$$

**U-Substitution** When thinking of doing a u-substitution on an integral remember:

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1. Exhaust all your possibilities before attempting a u-substitution; Use algebra or identities to simplify the integrand first (if possible), then try to integrate with the rules above.
2. If step 1 does not work or , pick u to be function whose is operated on (i.e. A function you are taking a square root, you are taking a log of, etc .). Or pick u to be the function whose derivative appears to be in the integrand.

**Area**

If trying to approximate the area under a function \( f(x) \) on the interval \([a, b]\) with \( n \) intervals and if using:

*Left End points* - the area approximately is:

\[
L_n = \sum_{k=1}^{n} f(a + (k - 1) \frac{b-a}{n}) \frac{b-a}{n}
\]

*Right End points* - the area approximately is:

\[
R_n = \sum_{k=1}^{n} f(a + k \frac{b-a}{n}) \frac{b-a}{n}
\]

*Midpoints* - the area approximately is:

\[
M_n = \sum_{k=1}^{n} f(a + (k - \frac{1}{2}) \frac{b-a}{n}) \frac{b-a}{n}
\]

**Fundamental Theorem of Calculus I** Assume \( f(x) \) is continuous on \([a,b]\) and let \( F(x) \) be an antiderivative of \( f(x) \) (i.e. \( F'(x) = f(x) \)) on \([a,b]\). Then

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\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

**Fundamental Theorem of Calculus II** Assume \( f(x) \) is continuous on an interval, and let \( a \) be any point in that interval. Define the function \( F(x) \) as follows:

\[ F(x) = \int_{a}^{x} f(t) \, dt \]

Then \( F'(x) = f(x) \) for all \( x \) in the interval.

**Volume of a solid rotated about the x axis (disk method)** If \( f(x) \) is continuous and \( f(x) \geq 0 \) on \([a,b]\), then the volume \( V \) obtained by rotating the region under the graph about the x-axis is:

\[ V = \pi \int_{a}^{b} [f(x)]^2 \, dx \]

**Volume of a solid rotated about the y axis (disk method)** If \( f(y) \) is continuous and \( f(y) \geq 0 \) on \([c,d]\), then the volume \( V \) obtained by rotating the region under the graph about the y axis is:

\[ V = \pi \int_{c}^{d} [f(y)]^2 \, dy \]

**Volume of a solid rotated about the y axis (shell method)** If \( f(x) \) is continuous and the volume \( V \) of the solid obtained by rotating the region under the graph of \( y = f(x) \) over the interval \([a, b]\) about the y- axis is equal to:

\[ V = 2\pi \int_{a}^{b} xf(x) \, dx \]
Volume of a solid rotated about they x axis (shell method) If \( f(y) \) is continuous and the volume \( V \) of the solid obtained by rotating the region under the graph of \( x = f(y) \) over the interval \([c, d]\) about the x-axis is equal to:

\[
V = 2\pi \int_c^d yf(y) \, dy
\]

Midpoint Rule The Nth Midpoint Approximation to \( \int_a^b f(x) \, dx \) is

\[
M_N = \Delta x \left( f(c_1) + f(c_2) + \cdots + f(c_N) \right)
\]

where \( \Delta x = \frac{b-a}{N} \) and \( c_k = a + (j - \frac{1}{2})\Delta x \) is the midpoint of the \( j \) th interval \([x_{j-1}, x_j]\).

Trapezoid Rule The Nth trapezoidal approximation to \( \int_a^b f(x) \, dx \) is

\[
T_N = \frac{1}{2} \Delta x (y_0 + 2y_1 + \cdots + 2y_{N-1} + y_N)
\]

where \( \Delta x = \frac{b-a}{N} \) and \( y_j = f(a + j\Delta x) \)

Simpson’s Rule Assume that \( N \) is even. Let \( \Delta x = \frac{b-a}{N} \) and \( y_j = f(a + j\Delta x) \).

The Nth approximation to \( \int_a^b f(x) \, dx \) by Simpson’s Rule is the quantity

\[
S_N = \frac{1}{3} \Delta x [y_0 + 4y_1 + 2y_2 + \cdots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N]
\]

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**Error Bounds for** \( T_N, M_N \) **and** \( S_N \)  
Let \( K_2 \) be a number such that \(|f''(x)| \leq K_2\) on 
\([a,b]\) and \( K_4 \) be a number such that \(|f^{(4)}(x)| \leq K_4\) for all \( x \in [a,b]\). Then we have the following error bounds for \( T_N, M_N \) and \( S_N \).

\[
\text{Error } (T_N) \leq \frac{K_2(b-a)^3}{12N^2} \quad \text{Error } (M_N) \leq \frac{K_2(b-a)^3}{24N^2} \quad \text{Error } (S_N) \leq \frac{K_4(b-a)^5}{180N^5}
\]

**Arclength** given a function \( y = f(x) \) from \( x = a \) to \( x = b \)

\[
\int_a^b \sqrt{1 + [f'(x)]^2} \, dx
\]

**Arclength** given a parametric functions, \( x(t) \) and \( y(t) \) from \( t = a \) to \( t = b \)

\[
\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt
\]

**Surface Area** Give a function \( f(x) \). If a region under \( f(x) \) from \( x = a \) to \( x = b \) is rotated around the \( x \) axis, the following formula will give you the surface area:

\[
\int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx
\]
We can also rotate a surface about a $y$ axis with given function $x = f(y)$ with a similar formula.

**Position, velocity, speed, acceleration, distance, and displacement:**

The following is a relationship between position ($s(t)$), velocity ($v(t)$), and acceleration ($a(t)$):

\[ v(t) = s'(t) \quad \text{or} \quad \int v(t)\,dt = s(t) + C \]

\[ a(t) = v'(t) = s''(t) \quad \text{or} \quad \int a(t)\,dt = v(t) + C \]

*Speed* is defined to be $|v(t)| = |s'(t)|$

*Displacement* from $t = a$ to $t = b$ is given by: $\int_a^b v(t)\,dt$

*Distance* traveled from $t = a$ to $t = b$ is given by: $\int_a^b |v(t)|\,dt$

A particle is **slowing down** on an interval if $v(t)$ and $a(t)$ have *opposite signs*.

A particle is **speeding up** on an interval if $v(t)$ and $a(t)$ have the *same sign*.  

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